

Testing SDLCQ in 2+1 dimensions

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The SDLCQ regularization is known to explicitly preserve supersymmetry in 1+1 dimensions. To test this property in higher dimensions, we consider supersymmetric Yang-Mills theory on $R \times S^1 \times S^1$. In particular, we choose one of the compact directions to be light-like and another to be space-like. This theory is totally finite, and thus we can solve for bound state wave functions and masses numerically without renormalizing. We present the masses as functions of the longitudinal and transverse resolutions and show that the masses converge rapidly in both resolutions. We study the behavior of the spectrum as a function of the coupling and find that at strong coupling there is a stable, well-defined spectrum which we present. We discuss also the massless spectrum and find several unphysical states that decouple at large transverse resolution.

1 Introduction

The motivations to consider $\mathcal{N} = 1$ supersymmetric Yang-Mills theories in 2+1 dimensions are manifold. For one, there is recent progress in understanding the properties of strongly coupled gauge theories with supersymmetry², some of which are believed to be interconnected through a web of strong-weak coupling dualities. There is a need to study these issues at a fundamental level. Ideally, we would like to solve for the bound states of these theories directly, and at any coupling. However, solving a field theory from first principles is typically an intractable task. Nevertheless, it has been known for some time that 1 + 1 dimensional field theories can be solved from first principles via a straightforward application of DLCQ³. Recently, a large class of supersymmetric gauge theories in two dimensions was studied using a supersymmetric form of DLCQ (SDLCQ), which is known to preserve supersymmetry at every stage of the calculation^{7,4}. It turns out that this formalism can be applied to higher-dimensional theories⁵. This is interesting, because in higher dimensions, due to the additional scale, theories have the potential of exhibiting a complex phase structure, which may include a strong-weak coupling duality.

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2 SDLCQ

We consider a three dimensional $SU(N_c)$ $\mathcal{N}=1$ super-Yang-Mills theory compactified on the space-time $\mathbf{R} \times S^1 \times S^1$. The calculations are performed in the large N_c limit. In particular, we use light-cone coordinates $x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1)$, compactify x^- on a light-like circle *a la* DLCQ, and wrap the remaining transverse coordinate x^\perp on a spatial circle. We are able to solve for bound state wave functions and masses numerically by diagonalizing the discretized light-cone supercharge. This procedure preserves supersymmetry at every step. The action of $\mathcal{N} = 1$ SYM(2+1) is

$$S = \int d^2x \int_0^L dx_\perp \text{tr} \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + i\bar{\Psi} \gamma^\mu D_\mu \Psi \right).$$

We decompose the spinor Ψ in terms of chiral projections ψ, χ and choose the light-cone gauge $A^+ = 0$. We solve for the non-dynamical fields A^- and χ and formulate the momentum operators in the physical degrees of freedom ($\phi \equiv A^2$)

$$P^+ = \int dx^- \int_0^L dx_\perp \text{tr} \left[(\partial_- \phi)^2 + i\psi \partial_- \psi \right],$$

$$P^- = \int dx^- \int_0^L dx_\perp \text{tr} \left[-\frac{1}{2} J \frac{1}{\partial_-^2} J - \frac{i}{2} D_\perp \psi \frac{1}{\partial_-} D_\perp \psi \right].$$

The canonical commutation relations yield the supersymmetry algebra

$$\{Q^+, Q^+\} = 2\sqrt{2}P^+, \{Q^-, Q^-\} = 2\sqrt{2}P^-, \{Q^+, Q^-\} = -4P_\perp.$$

We use the standard decomposition of the fields $\phi_{ij}(x^-, x^\perp)$ and $\psi_{ij}(x^-, x^\perp)$ into momentum modes $a_{ij}^\dagger(\underline{k})$ and $b_{ij}^\dagger(\underline{k})$, respectively. We defined $\underline{k} \equiv (k^+, n^\perp)$ for convenience. The (anti)commutation relations

$$\left[a_{ij}(\underline{k}), a_{lk}^\dagger(\underline{k}') \right] = \left\{ b_{ij}(\underline{k}), b_{lk}^\dagger(\underline{k}') \right\} = \delta(k^+ - k'^+) \delta_{n^\perp, n'^\perp} \delta_{il} \delta_{jk}.$$

yield the explicit form of the supercharges, which are listed in Ref. ¹. For the present discussion it suffices to know that the structure of Q^- is

$$Q^- = \frac{2^{3/4}\pi i}{L} \sum_{n^\perp \in \mathbf{Z}} \int_0^\infty dk^+ \frac{n^\perp}{\sqrt{k^+}} \left[a_{ij}^\dagger(\underline{k}) b_{ij}(\underline{k}) - b_{ij}^\dagger(\underline{k}) a_{ij}(\underline{k}) \right] + g\tilde{Q}^-,$$

where \tilde{Q}^- contains the terms with three operators. We note also that the supercharge Q^- is linear in the coupling g , and thus the Hamiltonian P^- is quadratic in g .

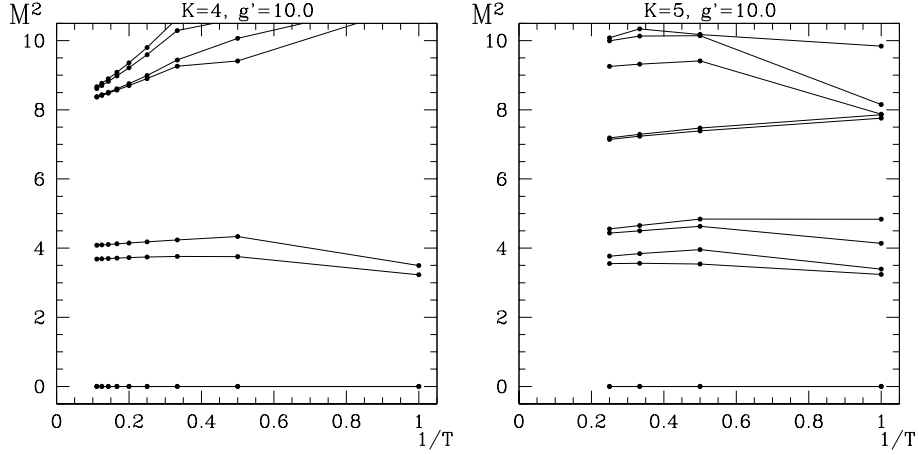


Figure 1: Plot of bound state mass squared M^2 in units of $2\pi^2/L^2$ as a function of the transverse resolution T for a coupling $g' = 10$ and for longitudinal resolutions $K = 4$ (a) and $K = 5$ (b). Boson and fermion masses are identical.

We now perform the truncation procedure. The harmonic resolution K plays the role of a longitudinal cutoff as usual, and longitudinal momentum fractions take values $\frac{k_i^+}{P^+} = \frac{n_i}{K}$, $n_i = 1, 2, \dots, K$. The transverse cutoff T allows for momenta $k_i^\perp = 2\pi n_i^\perp/L$ with $n_i^\perp = 0, \pm 1, \pm 2, \dots, \pm T$. This prescription preserves parity symmetry in transverse directions. How does such a truncation affect the supersymmetry properties of the theory? Note first that an operator relation $[A, B] = C$ in the full theory is not expected to hold in the truncated formulation. However, if A is quadratic in terms of fields (or in terms of creation and annihilation operators), one can show that the relation $[A, B] = C$ implies $[A_{tr}, B_{tr}] = C_{tr}$ for the truncated operators A_{tr}, B_{tr} , and C_{tr} . In our case, Q^+ is quadratic, and so the relations $\{Q_{tr}^+, Q_{tr}^+\} = 2\sqrt{2}P_{tr}^+$ and $\{Q_{tr}^+, Q_{tr}^-\} = 0$ are true in the $P_\perp = 0$ sector of the truncated theory. The anticommutator $\{Q_{tr}^-, Q_{tr}^-\}$, however, is not equal to $2\sqrt{2}P_{tr}^-$. So the diagonalization of $\{Q_{tr}^-, Q_{tr}^-\}$ will yield a different bound-state spectrum than the one obtained after diagonalizing $2\sqrt{2}P_{tr}^-$. Of course, the two spectra should agree in the limit $T \rightarrow \infty$. At any finite truncation, however, we have the liberty to diagonalize either of these operators. The choice of $\{Q_{tr}^-, Q_{tr}^-\}$ specifies our regularization scheme. Choosing to diagonalize the light-cone supercharge Q_{tr}^- has an important advantage: *the spectrum is exactly supersymmetric for any truncation*. In contrast, the spectrum of the Hamiltonian P_{tr}^- becomes supersymmetric only in the infinite resolution limit.

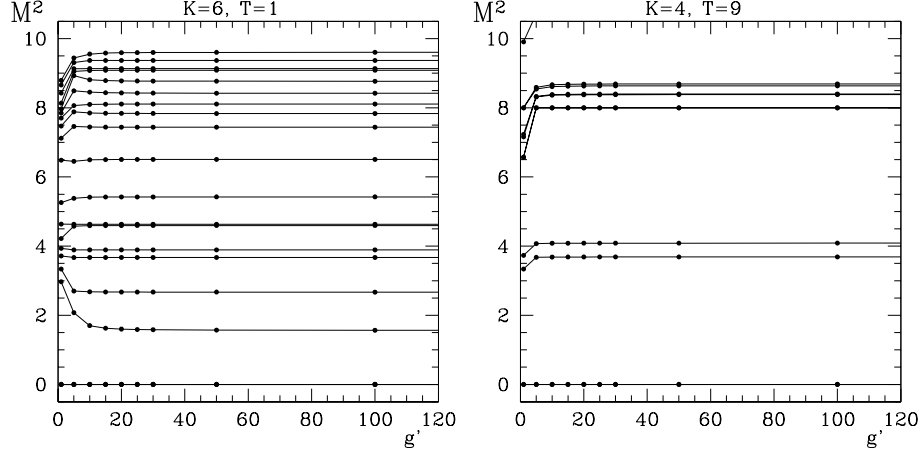


Figure 2: Plot of bound state mass squared M^2 in units of $2\pi^2/L^2$ as a function of the coupling g' . We show the plots for $K = 4, T = 9$ (a) and $K = 6, T = 1$ (b).

Let us take a look at the discrete symmetries of Q^- . The three commuting symmetries Z_2 are parity in the transverse direction

$$P : a_{ij}(k, n^\perp) \rightarrow -a_{ij}(k, -n^\perp), \quad b_{ij}(k, n^\perp) \rightarrow b_{ij}(k, -n^\perp),$$

which anti-commutes with Q^+ and P_\perp , and a generalized T symmetry

$$S : a_{ij}(k, n^\perp) \rightarrow -a_{ji}(k, n^\perp), \quad b_{ij}(k, n^\perp) \rightarrow -b_{ji}(k, -n^\perp).$$

To close the group, we need a third symmetry, namely $R = PS$. An interesting detail of the symmetry considerations is the fact that the P symmetry leads to exactly degenerate eigenvalues. This means in turn that all massive eigenvalues are four-fold degenerate. The argument goes as follows. Start with a massive state with positive parity $|M+\rangle$ which obeys

$$(Q^-)^2|M+\rangle = M^2|M+\rangle, \quad P|M+\rangle = +|M+\rangle.$$

Then $Q^+Q^-|M+\rangle$ is a state with same mass but opposite parity

$$PQ^+Q^-|M+\rangle = -Q^+Q^-P|M+\rangle = -Q^+Q^-|M+\rangle.$$

3 Numerical Results

With the truncation prescription described above, we can solve the discretized eigenvalue problem

$$2P^+P^-|\psi\rangle = M^2|\psi\rangle,$$

characterized by the cutoffs K and T , on the computer. This is equivalent to constructing the supercharge Q^- in the usual Fock basis, and then diagonalizing it. If the resulting mass (squared) eigenvalues M^2 are plotted as a function of the dimensionless coupling $g' = g\sqrt{NL}/4\pi^3$, several striking features emerge. Namely, as was noted in Ref. ⁵, one finds a very stable strong-coupling spectrum. Secondly, we find states which fall off fast to zero mass with increasing coupling. Since the previous work ⁵ was a calculation of the spectrum at $T \equiv 1$, it is natural to ask whether the well defined large g' spectrum survives at $T \rightarrow \infty$, and to study the properties of the states with masses decreasing at large g' . A further question is if the number of massless states is independent of the transverse cutoff T .

Our previous SDLCQ calculations were done using a code written in Mathematica and performed on a PC. This code has now been rewritten in C++ and some of the present work was done on supercomputers. We were able to perform numerical diagonalizations for $K = 2$ through 7 and for values of T up to $T = 9$ at $K = 4$ and $T = 1$ at $K = 7$.

Massive spectrum: Little is known about the large coupling spectrum of quantum field theories, with the exception of theories in 1+1 dimensions. There, however, the concept of large coupling has no meaning, since the coupling is only a multiplicative constant in the Hamiltonian. In particular, there is no weak/strong duality known in $\mathcal{N} = 1$ SYM(2+1), which could give some clue how the spectrum looks like. We performed therefore a non-perturbative calculation in SDLCQ to directly access the spectrum.

In Fig. 2 we plot the bound state masses squared M^2 as a function of the transverse resolution T for $K = 4$ and $K = 5$. We see that the curves are very flat, thus exhibiting fast convergence in transverse cutoff T . The continuum result can be obtained by extrapolating the curves to $1/T \rightarrow 0$. Let us look at the bound states as a function of the coupling g' , focusing on the large coupling regime. We see, Fig. 2, that the states are extremely stable in g' , *i.e.* they are quasi independent of coupling. We find this behavior at every value of K , and even irrespective of the value of the transverse cutoff T . We show the bound state mass as a function of $1/K$ in Fig. 3(a). These results are the first calculation of the strong-coupling bound states of $\mathcal{N} = 1$ SYM in $2 + 1$ dimensions. As we increase the resolution we are able to see states that have, as their primary component, more and more partons, and, as we have seen in other supersymmetric theories, many of these states appear at low energies. This accumulation of high-multiplicity low-mass states appears to be a unique property of SUSY theories. In non-SUSY theories the new states appear at increasing energies. In the dimensionally reduced version of this theory we saw that the accumulation point of these low-mass states appeared to be at

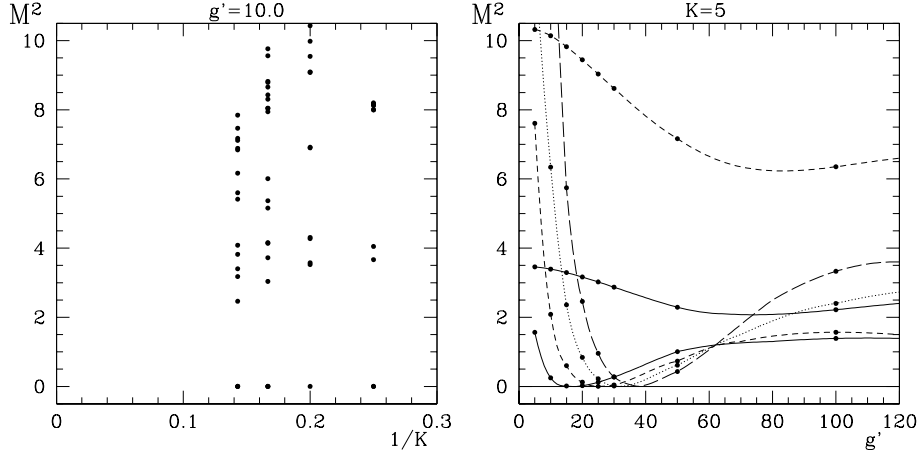


Figure 3: Plot of bound state mass squared M^2 in units of $2\pi^2/L^2$ as a function of $1/K$ for coupling $g' = 10$ (a). States with masses falling rapidly with increasing coupling as a function of g' (b) at the transverse resolutions $T = 1$ (solid lines), $T = 2$ (dashed lines), $T = 3$ (short dashed line) and $T = 4$ (long dashed line).

zero mass^{5,6}. Here again we see clear evidence of an accumulation of low mass states, however we don't have sufficient information to say whether an accumulation point exists.

Massless states: There are two kinds of massless states in the spectrum. Firstly, the states massless for $g' \rightarrow 0$. They are massless because at small coupling, only first term of the supercharge, Eq. (1), gives a contribution. Then all partons with $n^\perp = 0$ (anti-)commute with Q^- . Thus all states made out of these partons are massless, *and* massless states contain just these partons. The set of massless states at $g' = 0$ therefore coincides with a Hilbert space of the theory dimensionally reduced to $1+1$. Moreover, the whole infrared spectrum of SYM2 + 1 at small coupling is governed by the dimensionally reduced theory (see⁵ for details) Secondly, we see states that are exactly massless at any coupling. These are $2(K-1)$ BPS states, fulfilling $Q^-|m=0\rangle = 0$, $Q^+|m=0\rangle \neq 0$. It is therefore easy to construct the massless states of (2+1) theories, at least at large N_c .

Unphysical states: Finally, we can unambiguously detect unphysical states due to their special properties. Namely, these states vary strongly with coupling g' and they appear (predominantly) for K odd and decouple for $T \rightarrow \infty$, see Fig. 3. It is thus easy to classify and to remove the unphysical states from the spectrum.

4 Conclusions

We have shown that the SDLCQ formalism naturally extends to higher dimensions. Rapid convergence in transverse direction is found. Concerning the specific theory, we obtained the strong coupling spectrum of $\mathcal{N} = 1$ SYM(2+1). The bound states are extremely stable for $g' \rightarrow \infty$. The unphysical states in the spectrum can be identified and removed. We found no new massless states at strong coupling compared to previous work⁵. The massless sector of the theory is completely determined by the dimensional reduced model. The light states turn out to be string-like and might contain physics of dual theories. Also, the theory might be conformal at decompactification limit. We are currently working on analytical and numerical improvements of the approach. We expect to be able to address problems like the very interesting $\mathcal{N} = 4$ SYM in 3+1 and $\mathcal{N} = 1$ SYM(2+1) including a Chern-Simons term conjectured to break supersymmetry, in the near future.

Acknowledgments

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